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Abstract: We give a definition of the Delaunay triangulation of a point set in a closed Euclidean d-manifold, i.e. a compact quotient space of the Euclidean space for a discrete group of isometries (a so-called Bieberbach group or crystallographic group). We describe a geometric criterion to check whether a partition of the manifold actually forms a triangulation (which subsumes that it is a simplicial complex). We provide an algorithm to compute the Delaunay triangulation of the manifold defined by a given set of points, if it exists. Otherwise, the algorithm returns the Delaunay triangulation of a finitely sheeted covering space of the manifold.

Whereas there was prior work for the special case of the flat torus, as far as we know this is the first result for general closed Euclidean d-manifolds. This research is motivated by application fields, like computational structural biology for instance, showing a need to perform simulations in quotient spaces of the Euclidean space by more general groups of isometries than the groups generated by d independent translations.

Key-words: Delaunay triangulation, flat manifold, Euclidean manifold, closed manifold, crystallographic groups, simplicial complex

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Triangulations de Delaunay d'ensembles de points dans des variétés euclidiennes fermées de dimension d

Résumé : Nous donnons une définition de la triangulation de Delaunay d'un ensemble de points dans une variété euclidienne fermée de dimension d, c'est-à-dire un quotient compact de l'espace Euclidien par l'action d'un groupe discret d'isométries (dit aussi groupe de Bieberbach ou groupe cristallographique). Nous décrivons un critère geométrique permettant de vérifier si une partition de la variété forme réellement une triangulation (ce qui sous-entend que c'est un complexe simplicial). Nous proposons un algorithme calculant, si elle existe, la triangulation de Delaunay de la variété, définie par un ensemble de points donné. Sinon, l'algorithme fournit la triangulation de Delaunay d'un revêtement fini de la variété.

Alors qu'il existe des travaux antérieurs pour le cas particulier du tore plat, à notre connaissance c'est le premier résultat pour des variétés euclidiennes fermées générales de dimension d. Cette recherche est motivée par des domaines d'application tels que la modélisation moléculaire, par exemple, qui expriment le besoin d'effectuer des simulations dans des espèces quotients de l'espace euclidien par des groupes d'isométries plus généraux que les groupes engendrés par d translations indépendantes.

Mots-clés : triangulation de Delaunay, variété plate, variété euclidienne, variété fermée, groupes cristallographiques, complexe simplicial

1 Introduction

The Delaunay triangulation of a point set in $\mathbb{E}^d$ is a well-studied structure in computational geometry. Efficient algorithms are known and there exist various implementations. We extend the well-known incremental algorithm [Bow81] which computes the Delaunay triangulation in $\mathbb{E}^d$ to the case of closed Euclidean d-manifolds. Closed Euclidean d-manifolds can be represented as quotient spaces of $\mathbb{E}^d$ for a certain class of discrete groups of isometries, the so-called Bieberbach groups or crystallographic groups [Thu97a]. In such manifolds, there are sets of points that do not define a Delaunay triangulation. We describe a geometric test that can be used to check this during the incremental construction of the triangulation. In such cases, the algorithm actually computes the Delaunay triangulation of copies of the input points in a finitely sheeted covering space of the manifold.

This paper is a generalization of [CT09], which discussed the case of the three-dimensional flat torus $\mathbb{T}^3$, and which was accompanied by a CGAL software package [CT10]. The flat torus is the quotient space of $\mathbb{E}^3$ by a group of three independent translations. While this case fulfills the needs of many application fields, some of them, like computational biology[1], require more general manifolds, that are quotient spaces of $\mathbb{E}^3$ by other crystallographic groups.

We summarize the work done on the flat torus in Section 3. In Section 4, we introduce closed Euclidean d-manifolds and their properties. Section 5 studies Delaunay triangulations in closed Euclidean d-manifolds and shows using the Bieberbach theorem that there is always a finitely-sheeted covering space of the manifold, in which the Delaunay triangulation is defined for any set of points. Section 6 proposes an algorithm.

2 Preliminaries

Let us briefly recall a few elementary definitions.

A *k-simplex* σ in $\mathbb{E}^d$ ($k \leq d$) is the convex hull of $k+1$ affinely independent points $\mathcal{P}_\sigma = \{p_0, p_1, \ldots, p_k\}$. A simplex τ defined by $\mathcal{P}_\tau \subseteq \mathcal{P}_\sigma$ is a *face* of σ and has σ as a *coface*. This is denoted by $\sigma \geq \tau$ and $\tau \leq \sigma$.

In this paper, we consider triangulations with an infinite number of vertices, so, we use the following definition of a simplicial complex:

Definition 1 ([Lee00]) *A* simplicial complex *is a set $\mathcal{K}$ of simplices such that:*

(i). $\sigma \in \mathcal{K}, \tau \leq \sigma \Rightarrow \tau \in \mathcal{K}$

(ii). $\sigma, \sigma' \in \mathcal{K} \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$

(iii). Every point in a simplex of $\mathcal{K}$ has a neighborhood that intersects at most finitely many simplices in $\mathcal{K}$ (local finiteness).

Note that the previous definition is completely combinatorial; with an appropriate definition of a simplex, it remains valid in any topological space $\mathbb{X}$. We will propose a definition of a simplex adapted to the case of a closed Euclidean manifold in Section 5.

[1] see for instance the talk *Computational Structural Biology: Periodic Triangulations for Molecular Dynamics*, by Julie Bernauer, at the workshop *Subdivide and tile*, `http://www-sop.inria.fr/geometrica/collaborations/OrbiCG/program.html`

A *triangulation* of a topological space $\mathbb{X}$ is a simplicial complex $\mathcal{K}$ such that $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$ is homeomorphic to $\mathbb{X}$. A triangulation is defined by a point set $\mathcal{P}$ if its set of vertices (0-simplices) is identical to $\mathcal{P}$.

A triangulation of a point set $\mathcal{P}$ in $\mathbb{E}^d$ is a *Delaunay* triangulation *iff* each simplex satisfies the *Delaunay property*, i.e. its circumscribing ball does not contain any point of $\mathcal{P}$ in its interior [BY98, dBvKOS00].

3 Review of Triangulations in $\mathbb{T}^3$

Let $\mathcal{P}$ be a finite point set in the three-dimensional Euclidean space $\mathbb{E}^3$. The three-dimensional *flat torus* $\mathbb{T}^3$ is defined as the quotient space $\mathbb{E}^3/\mathcal{G}$, where $\mathcal{G} = (\mathbb{Z}^3, +)$ or equivalently the group generated by the three orthogonal unit translations of $\mathbb{E}^3$. Intuitively, $\mathbb{T}^3$ is obtained by identifying all three pairs of opposite facets of a cube of $\mathbb{E}^3$.

Let $\pi : \mathbb{E}^3 \to \mathbb{T}^3$ denote the quotient map and $DT(\mathcal{GP})$ denote the Delaunay triangulation of the infinite point set $\mathcal{GP} = \{p + z \mid p \in \mathcal{P}, z \in \mathbb{Z}^3\}$ in $\mathbb{E}^3$.

Definition 2 ([CT09]) *If $\pi(DT(\mathcal{GP}))$ is a simplicial complex in $\mathbb{T}^3$, then we call it the* Delaunay triangulation of $\pi(\mathcal{P})$ in $\mathbb{T}^3$.

There are point sets in $\mathbb{T}^3$ that do not define a Delaunay triangulation: For example, if $\mathcal{P}$ consists of one point only, then $\pi(DT(\mathcal{GP}))$ is not a simplicial complex: the condition (ii) of Definition 1 is violated. Let us recall that the *1-skeleton* of a simplicial complex is the graph that consists of all edges and vertices.

Theorem 1 ([CT09]) *If the 1-skeleton of $\pi(DT(\mathcal{GP}))$ does not contain cycles of length ≤ 2, then $\pi(DT(\mathcal{GP}))$ is a triangulation of $\mathbb{T}^3$.*

This yields a geometric criterion for $\pi(DT(\mathcal{GP}))$ to be a triangulation. Note that it also holds for supersets of $\mathcal{P}$, which will be useful for the algorithm later on.

Corollary 2 ([CT09]) *Let $\mathbb{B}$ denote the largest 3-ball in $\mathbb{E}^3$ that does not contain points of $\mathcal{GP}$ in its interior. If $\mathbb{B}$ has diameter $< \frac{1}{2}$, then $\pi(DT(\mathcal{GP}'))$ is a triangulation in $\mathbb{T}^3$ for any finite $\mathcal{P}' \supseteq \mathcal{P}$.*

Consider the quotient space $\mathbb{T}^3_{27} := \mathbb{E}^3/\mathcal{G}_{27}$ with $\mathcal{G}_{27} := ((3\mathbb{Z})^3, +)$. Then $\mathbb{T}^3_{27}$ is a 27-sheeted covering space of $\mathbb{T}^3$ (see e.g. [Arm82] for a discussion on covering spaces). The following theorem ensures that, for any set of points, it is always possible to compute a Delaunay triangulation in $\mathbb{T}^3_{27}$.

Theorem 3 ([CT09]) *For any finite point set $\mathcal{P}$ in $\mathbb{E}^3$, the projection of the Delaunay triangulation of $\mathcal{GP}$ in $\mathbb{E}^3$ onto $\mathbb{T}^3_{27}$ is a triangulation.*

Theorem 3 and Corollary 2 lead to a modified version of the incremental algorithm for computing Delaunay triangulations in $\mathbb{E}^3$ [Bow81]:

- It starts with inserting 27 copies per input point, computing their Delaunay triangulation in $\mathbb{T}^3_{27}$.

- Once the largest 3-ball not containing any vertex in its interior has diameter smaller than $\frac{1}{2}$, the algorithm switches to computing in $\mathbb{T}^3$ and inserts each of the remaining points only once.

While computing in $\mathbb{T}^3_{27}$, 27 copies of points of $\mathcal{P}$ are inserted one by one. So, in fact the following extended version of Theorem 3 is needed for the algorithm to work:

Theorem 4 ([CT09]) *Theorem 3 still holds if we replace $\mathcal{GP}$ by $\mathcal{GP} \cup \mathcal{G}_{27}Q$ for any $Q \subseteq \mathcal{G}p$ with any $p \in \mathbb{E}^3$.*

As shown in [CT09], depending on the set of points $\mathcal{P}$, the algorithm computes either a triangulation of $\mathbb{T}^3$ if possible, or a triangulation of $\mathbb{T}^3_{27}$, which is homeomorphic to $\mathbb{T}^3$. In practice, data sets are likely to define a Delaunay triangulation of $\mathbb{T}^3$. The algorithm has optimal randomized worst-case complexity.

4 Closed Euclidean Manifolds

This section is dedicated to introducing closed Euclidean manifolds, their properties, and how to construct them. Most concepts mentioned in this section are taken from [Thu97b].

A *closed manifold* is a compact manifold without boundary. A d-manifold is called *Euclidean* or *flat*, if every point has a neighborhood isometric to a neighborhood in $\mathbb{E}^d$.

We need some more notions: Let $\mathcal{G}$ be a group and $\mathcal{H}$ denote a subgroup of $\mathcal{G}$. $\mathcal{H}$ is called *normal* in $\mathcal{G}$ if it is invariant under conjugation, i.e., if for all $h \in \mathcal{H}$ and $g \in \mathcal{G}$, $ghg^{-1} \in \mathcal{H}$. For a group element $g \in \mathcal{G}$, the set $\{gh \mid h \in \mathcal{H}\}$ is called a *coset* of $\mathcal{H}$ in $\mathcal{G}$. The *index of a subgroup* $\mathcal{H}$ in $\mathcal{G}$ is defined as the number of cosets of $\mathcal{H}$ in $\mathcal{G}$.

A d-dimensional *Bieberbach group* $\mathcal{G}_B$ is a discrete group of isometries of $\mathbb{E}^d$ with compact quotient space $\mathbb{E}^d/\mathcal{G}_B$. Such groups are also called *crystallographic groups* or *space groups*.

Theorem 5 (Bieberbach [Bie10])

- *Let $\mathcal{G}_B$ be a d-dimensional Bieberbach group. There is a group $\mathcal{G}_T$ of d linearly independent translations that is a normal subgroup of $\mathcal{G}_B$ of finite index. We call $\mathcal{G}_T$ the* translational subgroup *of $\mathcal{G}_B$.*

- *For any d, there is only a finite number of d-dimensional Bieberbach groups, up to isomorphism.*

Note that the quotient space $\mathbb{E}^d/\mathcal{G}_B$ is not necessarily a manifold: If $\mathcal{G}_B$ leaves points fixed, these points do not have a neighborhood in $\mathbb{E}^d/\mathcal{G}_B$ that is homeomorphic to a neighborhood in $\mathbb{E}^d$. The quotient space $\mathbb{E}^d/\mathcal{G}_B$ can always be described by the more general concept of an *orbifold* [BMP03, Thu02]. For the quotient space to be a manifold, the group must not have fixed points. In other words the group must be *torsion-free*, i.e., the identity must be the only element of finite order.

If $\mathcal{G}_T$ is a subgroup of d independent translations of $\mathcal{G}_B$, then $\mathbb{E}^d/\mathcal{G}_T$ is a d-torus.

Theorem 6 ([Thu97b]) *Any closed Euclidean d-manifold corresponds up to diffeomorphism to exactly one quotient space $\mathbb{E}^d/\mathcal{G}_B$, where $\mathcal{G}_B$ is a torsion-free d-dimensional Bieberbach group.*

This means that it is sufficient to consider torsion-free Bieberbach groups to completely classify closed Euclidean manifolds.

According to Theorem 5, there are only finitely many d-dimensional Bieberbach groups, up to isomorphism. In dimension 2 there are 17, in dimension 3 there are 230.[2] In two dimensions, there are only two torsion-free Bieberbach groups and thus two closed Euclidean manifolds, up to isomorphism: the torus and the Klein bottle. In three dimensions, there are 10 closed Euclidean manifolds, four of which are non-orientable.

5 Triangulations in Closed Euclidean Manifolds

The goal of this section is to generalize the main results of [CT09] recalled in Section 3.

Let $\mathcal{G}_F$ be a torsion-free d-dimensional Bieberbach group, $\mathcal{P}$ a finite point set in $\mathbb{E}^d$, $\mathbb{X} := \mathbb{E}^d/\mathcal{G}_F$ a closed Euclidean manifold with quotient map $\pi : \mathbb{E}^d \to \mathbb{X}$, and $DT(\mathcal{G}_F\mathcal{P})$ the Delaunay triangulation of the infinite point set $\mathcal{G}_F\mathcal{P}$ in $\mathbb{E}^d$.

To be able to consider triangulations of $\mathbb{X}$, as defined in Section 2, we first give a definition for a simplex in such a manifold. A similar definition can be found in [Wil08].

Definition 3 *Let $\mathcal{P}_\sigma$ be a set of $k+1$ ($k \leq d$) points in $\mathbb{E}^d$. If the restriction $\pi|_{\mathrm{Ch}(\mathcal{P}_\sigma)}$ of π to the convex hull $\mathrm{Ch}(\mathcal{P}_\sigma)$ of $\mathcal{P}_\sigma$ is injective, the image of $\mathrm{Ch}(\mathcal{P}_\sigma)$ by π is called a k-*simplex *in $\mathbb{X}$.*

Roughly speaking, this definition requires simplices not to self-intersect in the quotient manifold. We can now adapt Definition 2 to the Delaunay triangulation of $\pi(\mathcal{P})$ in $\mathbb{X}$:

Definition 4 *If $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is a triangulation of $\mathbb{X}$ (which subsumes that it is a simplicial complex in $\mathbb{X}$), then we call it the* Delaunay triangulation of $\pi(\mathcal{P})$ in $\mathbb{X}$.

For the discussions below we need the following two values:

1. The minimum distance $\delta(\mathcal{G})$ by which a group $\mathcal{G}$ moves a point:
$$\delta(\mathcal{G}) = \min_{p\in\mathbb{E}^d, g\in\mathcal{G}, g\neq 1_\mathcal{G}} \mathrm{dist}(p, gp),$$
where $1_\mathcal{G}$ denotes the unit element of $\mathcal{G}$. Note that if $\mathcal{G}$ is torsion-free and discrete, then $\delta(\mathcal{G}) > 0$ holds.

2. The diameter $\Delta(\mathcal{S})$ of the largest d-ball $\mathbb{B}$ in $\mathbb{E}^d$ that does not contain any point of a set $\mathcal{S}$ in its interior.

We now generalize Theorem 1:

[2]The number of Bieberbach groups by dimension is assigned the id A006227 in the On-Line Encyclopedia of Integer Sequences [Slo]. The number of *torsion-free* Bieberbach groups is assigned the id A059104.

Theorem 7 *If the 1-skeleton of $\pi(DT(\mathcal{G}_F\mathcal{P}))$ does not contain cycles of length ≤ 2 then $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is a triangulation of $\mathbb{X}$.*

Most parts of the proof of Theorem 1 are completely combinatorial and do not depend on the space, so they extend directly to $\mathbb{X}$ and we postpone them to the Appendix. We only prove the generalized version of Lemma 4.1 of [CT09]:

Lemma 8 *Let $\mathcal{K}$ be a set of simplices in $\mathbb{E}^d$ whose vertices are exactly the elements of $\mathcal{G}_F\mathcal{P}$, and that fulfills conditions (i) and (ii) of Definition 1, and the Delaunay property with respect to $\mathcal{G}_F\mathcal{P}$. Then $\mathcal{K}$ satisfies the local finiteness property (iii) as well. Thus, $\mathcal{K}$ is a simplicial complex.*

Proof. Let us assume that there is a vertex $v \in \mathcal{K}$ with an infinite number of incident simplices and thus an infinite number of incident edges in $\mathcal{K}$. Since $\mathcal{P}$ contains only a finite number of points, there must be at least one point q in $\mathcal{P}$ such that infinitely many points of the discrete point set $\mathcal{G}_F q$ are adjacent to v. Note that $\delta(\mathcal{G}_F) > 0$ and $\Delta(\mathcal{G}_F q) < \infty$ hold because $\mathcal{G}_F$ is a torsion-free Bieberbach group. Projecting all the edges from v to points in $\mathcal{G}_F q$ onto the unit d-sphere S centered in v yields an infinite point set $\mathcal{P}_S$. As S is bounded, $\mathcal{P}_S$ must have an accumulation point. We choose q_1 and q_2 from $\mathcal{G}_F q$ such that the distance between their projections onto S is smaller than ε for some $\varepsilon < \frac{\delta(\mathcal{G}_F)^3}{\Delta(\mathcal{G}_F q)^3}$ and $\varepsilon > 0$. Without loss of generality, we assume that $\mathrm{dist}(v, q_2) \geq \mathrm{dist}(v, q_1)$. We give a lower bound on the diameter D of the circumcircle of the triangle vq_1q_2 (see Fig. 1): D is given by the product of the three edge lengths divided by twice the triangle's area. The three edge lengths are each at least $\delta(\mathcal{G}_F)$. Also $\mathrm{dist}(v, q_2) \leq \Delta(\mathcal{G}_F q)$ and the height of the triangle corresponding to the segment vq_2 is at most $\Delta(\mathcal{G}_F q)\varepsilon$. This yields $D \geq \frac{\delta(\mathcal{G}_F)^3}{\Delta(\mathcal{G}_F q)^2 \varepsilon} > \frac{\delta(\mathcal{G}_F)^3}{\Delta(\mathcal{G}_F q)^2 \frac{\delta(\mathcal{G}_F)^3}{\Delta(\mathcal{G}_F q)^3}} = \Delta(\mathcal{G}_F q)$. So, the d-ball B_{vq_2} with v and q_2 on its boundary must have diameter larger than $\Delta(\mathcal{G}_F q)$ to not contain q_1 in its interior. As the largest empty d-ball has diameter $\Delta(\mathcal{G}_F q)$, B_{vq_2} cannot be empty, which is a contradiction to $\mathcal{K}$ having the Delaunay property.

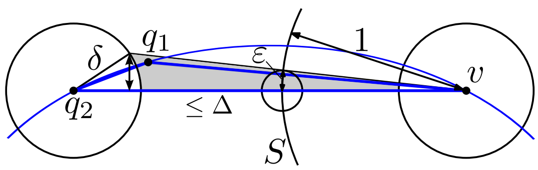

Figure 1: The gray area shows the possible positions of q_1

Let us now consider a point p in $\mathbb{E}^d$ that is not a vertex in $\mathcal{K}$. Let σ denote the simplex that contains p in its interior and let v_σ denote a vertex of σ. Let $St(v_\sigma)$ denote the set of simplices that v_σ is incident to. Above we have shown that $St(v_\sigma)$ contains only finitely many elements. The set $St(\sigma)$ of simplices that σ is incident to is a subset of $St(v_\sigma)$, thus it is finite. There is a neighborhood $U(p)$ that has non-empty intersection with exactly the elements $St(\sigma)$. □

Corollary 2 mentions the threshold $\frac{1}{2}$, which depends on the group $\mathcal{G}$. The generalized version of this corollary follows by simple geometric reasoning from Theorem 7.

Corollary 9 *If $\Delta(\mathcal{G}_F\mathcal{P}) < \frac{\delta(\mathcal{G}_F)}{2}$, then $\pi(DT(\mathcal{G}_F\mathcal{P}'))$ is a triangulation of $\mathbb{X}$ for any finite $\mathcal{P}' \supseteq \mathcal{P}$.*

For any torsion-free Bieberbach group there are point sets such that the condition of Corollary 9 is fulfilled, because δ is strictly positive and Δ can be made arbitrarily small by the choice of the point set.

Finally, we give a generalized version of Theorem 4.

Theorem 10 *There is a normal subgroup $\mathcal{G}_C$ of $\mathcal{G}_F$ of finite index such that the projection of the Delaunay triangulation of $\mathcal{G}_F\mathcal{P} \cup \mathcal{G}_C Q$ in $\mathbb{E}^d$ onto $\mathbb{X}_C = \mathbb{E}^d/\mathcal{G}_C$ is a triangulation for any finite point set $\mathcal{P}$ in $\mathbb{E}^d$ and any $Q \subseteq \mathcal{G}_F q$ with any $q \in \mathbb{E}^d$.*

Proof. According to Theorem 5, there is a group $\mathcal{G}_T$ of d linearly independent translations that is a normal subgroup of $\mathcal{G}_F$ with finite index h'. We choose generators $g_1, \ldots, g_d$ of $\mathcal{G}_T$ in the following way: Let g_1 be the shortest translation in $\mathcal{G}_T$. Let g_{i+1} be the shortest translation in $\mathcal{G}_T$ that is linearly independent of the translations $g_1, \ldots, g_i$. Note that $\Delta(\mathcal{G}_T p)$ does not depend on a specific choice of p and thus can be considered constant. We can find an integer coefficient c such that for each g_i the inequality $\mathrm{dist}(p, g_i^c p) > 2\Delta(\mathcal{G}_T p)$ holds for any $p \in \mathbb{E}^d$. The group $\mathcal{G}_C$ generated by $g_1^c, \ldots, g_d^c$ is a subgroup of $\mathcal{G}_T$ of index c^d with the property $\delta(\mathcal{G}_C) > 2\Delta(\mathcal{G}_T p)$ for any $p \in \mathbb{E}^d$. As $\mathcal{G}_T$ is normal in $\mathcal{G}_F$ we have $g g_T g^{-1} \in \mathcal{G}_T$ for each $g \in \mathcal{G}_F, g_T \in \mathcal{G}_T$. By construction of $\mathcal{G}_C$ there is a bijection between the $g_T \in \mathcal{G}_T$ and the $g_C \in \mathcal{G}_C$ given by $g_C = g_T^c$. Now it is easy to see that $\mathcal{G}_C$ is a normal subgroup of $\mathcal{G}_F$ with index $h = h' \cdot c^d$. Note that $\Delta(\mathcal{G}_C\mathcal{G}_F\mathcal{P}) = \Delta(\mathcal{G}_F\mathcal{P}) \leq \Delta(\mathcal{G}_T p)$ for any $p \in \mathbb{E}^d$. Thus $\Delta(\mathcal{G}_C\mathcal{G}_F\mathcal{P}) < \frac{\delta(\mathcal{G}_C)}{2}$ holds and according to Corollary 9 the projection of the Delaunay triangulation of $\mathcal{G}_C\mathcal{G}_F\mathcal{P} = \mathcal{G}_F\mathcal{P}$ onto $\mathbb{X}_C$ forms a triangulation, which remains true even when adding further points. $\square$

Note that the proof is constructive, i.e. it describes how to construct $\mathcal{G}_C$ from $\mathcal{G}_T$. The group $\mathcal{G}_T$ can be constructed from $\mathcal{G}_F$, e.g. using the Reidemeister-Schreier algorithm [Sim94]. Theorem 10 means that there exists a space $\mathbb{X}_C$, in which the Delaunay triangulation of the point set $\pi(\mathcal{P})$ is defined. The space $\mathbb{X}_C$ is a covering space of $\mathbb{X}$ with a finite number of sheets [Arm82]. Theorem 10 can also be understood by constructing $\mathbb{X}_C$ from $\mathbb{X}$ directly, as follows.

Definition 5 *A* fundamental domain *for a discrete group $\mathcal{G}$ of isometries in $\mathbb{E}^d$ with quotient map $\pi : \mathbb{E}^d \to \mathbb{E}^d/\mathcal{G}$ is a closed and convex subset $D_\mathcal{G}$ of $\mathbb{E}^d$ such that*

- *$D_\mathcal{G}$ contains at least one point of the preimage by π of any point in $\mathbb{E}^d/\mathcal{G}$.*
- *If $D_\mathcal{G}$ contains more than one point of the same preimage, then all points of this preimage lie on the boundary of $D_\mathcal{G}$.*

For example the unit cube is a fundamental domain of $\mathbb{T}^3$ as defined in Section 3.

Each closed Euclidean d-manifold has a d-torus as covering space with a finite number of sheets. This follows from Theorem 5 as discussed above. A fundamental domain of the d-torus is a d-dimensional hyperparallelepiped. By gluing two of these hyperparallelepipeds together, we get a new covering space that is again a d-torus. We can construct $\mathbb{X}_C$ by gluing as many copies of the fundamental domain as necessary to fulfill the condition in Corollary 9, i.e., $\Delta(\mathcal{G}_C\mathcal{G}_F\mathcal{P}) = \Delta(\mathcal{G}_F\mathcal{P}) < \frac{\delta(\mathcal{G}_C)}{2}$. See Figure 2 for an illustration in two dimensions.

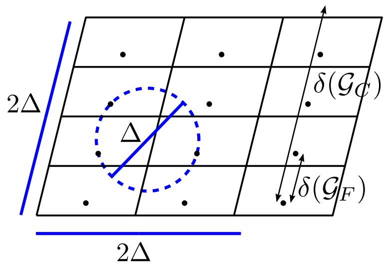

Figure 2: Sufficient number of copies of the fundamental domain

As an example we consider the flat Klein bottle $\mathbb{E}^2/\mathcal{G}_K$, where $\mathcal{G}_K$ is the group generated by a translation g_t and a glide-reflection g_g, that is a reflection together with a translation parallel to the reflection axis (see Figure 3). The group generated by g_t and g_g^2 is a translational subgroup of $\mathcal{G}_K$ of index 2. Now we can choose a subgroup of this translational subgroup with finite index that fulfills the condition of Theorem 10 as in Figure 2.

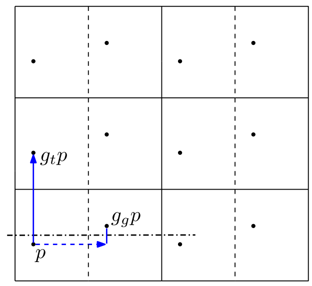

Figure 3: A part of the infinite point grid $\mathcal{G}_K p$.

Note that in both Corollary 9 and Theorem 10 we deal with the condition of the form $\Delta < \frac{\delta}{2}$. In Corollary 9 we adapt the point set $\mathcal{P}$ to decrease Δ, in Theorem 10 we adapt the group $\mathcal{G}_C$ to increase δ.

6 Algorithm

The algorithm described at the end of Section 3 generalizes to $\mathbb{X} = \mathbb{E}^d/\mathcal{G}_F$ using the results given in the previous section. The points of $\mathcal{P}$ are added one by one.

- The algorithm starts computing in the h-sheeted covering space $\mathbb{X}_C = \mathbb{E}^d/\mathcal{G}_C$ as in Theorem 10, inserting h copies per input point (here we call *copy* of a point p an element of its orbit by $\mathcal{G}_C$, i.e. a point $g_c p$, for $g_c \in \mathcal{G}_C$).
- Once the condition of Corollary 9 is met for the current point set, the algorithm switches to computing in $\mathbb{X}$ and continues to insert each of the remaining points only once.

If $\mathcal{P}$ is such that the condition of Corollary 9 is never fulfilled, then the algorithm returns the triangulation of the covering space $\mathbb{X}_C$.

Two issues appear, namely, how to store the current triangulation and how to insert a point.

The triangulation can be stored as a graph in the following way: Full-dimensional simplices are stored with a list of their vertices and neighbors. Each vertex contains the coordinates of the point it corresponds to. Additionally, each d-simplex stores the information on how to map it isometrically into $\mathbb{E}^d$, i.e. an appropriate element of the simplex' preimage under the quotient map π.

The point insertion can be split in a combinatorial and a geometric part. With the above data structure, the combinatorial part is the same as in the original incremental algorithm: the simplices that do not satisfy the Delaunay property after the insertion are removed, and the resulting hole is then triangulated by simplices incident to the new point. The geometric part mainly consists of computing the orientation of $d+1$ points (to locate the new point) and computing whether a point is situated inside or outside a d-ball circumscribing $d+1$ points (to check the Delaunay property). Both of these so-called *predicates* are evaluated in $\mathbb{E}^d$: For each d-simplex, on which we need to evaluate a predicate, we take its preimage under π from the data structure and evaluate the predicate on this preimage. This works exactly the same way even if $\mathbb{X}$ is non-orientable: The orientation of a preimage under π is computed using the orientation predicate in $\mathbb{E}^d$.

7 Conclusion

We extended the work of [CT09] to any closed Euclidean d-manifold.

A natural question is how the extend the results to general orbifolds. The results of Section 5 exclude Bieberbach groups with fixed points. However, from the Bieberbach theorem we know that any orbifold has a finitely sheeted covering space that is a closed Euclidean manifold and on which our approach works. So, while the approach cannot compute a triangulation in the orbifold, it can always compute a Delaunay triangulation in that covering space.

Another question is whether Delaunay triangulations can be defined and computed in hyperbolic manifolds that are quotients of hyperbolic spaces $\mathbb{H}^d$ by a Fuchsian group. The question is natural and exciting from a mathematical point of view, since the groups are much richer than the crystallographic groups. Preliminary attempts show the question to be quite challenging even in the simplest case of a group of four hyperbolic translations in $\mathbb{H}^2$. This case would already find applications in fields as diverse as computer graphics [RJG10] and neuromathematics [CF09].

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Appendix: Proof of Theorem 7

As announced in the main part of the paper, the proof follows the proof of Theorem 1 given in [CT09] very closely.

We call *original domain* a subset of a fundamental domain that contains exactly one element of the preimage by π of any point in $\mathbb{E}^d/\mathcal{G}$. Let $\mathcal{D}$ be an original domain.

At first, we show that Definition 4 actually makes sense: We verify that the simplices "match" under π, i.e. that all copies of a simplex in $DT(\mathcal{G}_F\mathcal{P})$ are mapped onto the same simplex in $\mathbb{X}$ under π. Then we can show that $|\pi(DT(\mathcal{G}_F\mathcal{P}))|$ is homeomorphic to $\mathbb{X}$. We also prove that if $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is a set of simplices, then it fulfills conditions (i) and (iii). Finally, we discuss under which circumstances condition (ii) is fulfilled, which yields the sufficient condition on $\pi(DT(\mathcal{G}_F\mathcal{P}))$ to be a triangulation.

Let us start with the first lemma:

Lemma 11 *If the restriction of π to any simplex in $DT(\mathcal{G}_F\mathcal{P})$ is injective, then $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is a set of internally disjoint simplices in $\mathbb{X}$ that do not contain any point of $\pi(\mathcal{G}_F\mathcal{P})$ in their interior.*

Proof. Consider a d-simplex σ of $DT(\mathcal{G}_F\mathcal{P})$, whose vertices are a $(d+1)$-tuple of points $\mathcal{P}_\sigma \subset \mathcal{G}_F\mathcal{P}$. σ satisfies the Delaunay property, so all copies $\mathcal{G}_F\mathcal{P}_\sigma$ also have an empty circumscribing ball. This shows that all these copies form d-simplices of $DT(\mathcal{G}_F\mathcal{P})$. This can be ensured even in degenerate cases: If we handle degeneracies as in [DT03], then the Delaunay triangulation of a set of cospherical points only depends on some intrinsic ordering between them. By choosing an ordering that is compatible with $\mathcal{G}_F$, all copies of that point set are triangulated in the same way.

Followingly, π collapses precisely all the copies of σ onto its equivalence class in $\mathbb{X}$. As any lower-dimensional simplex in $DT(\mathcal{G}_F\mathcal{P})$ is incident to some d-simplex, and thus is defined by a subset of its vertices, the same holds for simplices of any dimension.

Now the projections under π of two internally disjoint k-dimensional simplices σ and τ in $DT(\mathcal{G}_F\mathcal{P})$ are either equal or internally disjoint for $k \geq 1$, due to the bijectivity of π between both simplices and their respective images. The same argument implies that the interior of a simplex cannot contain any vertex. $\square$

We observe that $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is finite: $DT(\mathcal{G}_F\mathcal{P})$ is locally finite (Lemma 8), i.e. the star of any vertex is finite. As $\mathcal{P}$ is finite, $\mathcal{G}_F\mathcal{P}$ is discrete and all d-simplices have a certain volume larger than some constant. Followingly, there are only finitely many d-simplices necessary to fill the original domain $\mathcal{D}$ and thus $\mathbb{X}$. Finitely many d-simplices have only finitely many faces so the overall number of simplices in $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is finite as well.

Since $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is a triangulation by definition, to prove that it is a triangulation of $\mathbb{X}$, it only remains to show:

Lemma 12 *$|\pi(DT(\mathcal{G}_F\mathcal{P}))|$ is homeomorphic to $\mathbb{X}$.*

Proof. By its construction $|DT(\mathcal{G}_F\mathcal{P})| = \mathbb{E}^d$ and π is surjective. Followingly, $\pi(|DT(\mathcal{G}_F\mathcal{P})|)$ is equal to $\mathbb{X}$. Then, the chain of equalities

$$\begin{aligned}\pi(|DT(\mathcal{G}_F\mathcal{P})|) &= \pi\left(\bigcup_{\sigma\in DT(\mathcal{G}_F\mathcal{P})}\sigma\right) \overset{(1)}{=} \pi\left(\bigcup_{\tau\in\pi(DT(\mathcal{G}_F\mathcal{P}))}\pi^{-1}(\tau)\right)\\ &\overset{(2)}{=} \bigcup_{\tau\in\pi(DT(\mathcal{G}_F\mathcal{P}))}\tau = \bigcup_{\sigma\in DT(\mathcal{G}_F\mathcal{P})}\pi(\sigma) = |\pi(DT(\mathcal{G}_F\mathcal{P}))|\end{aligned}$$

holds with the following arguments:

(1) This step just regroups the order of the simplices but does not change the set (cf. Lemma 11).

(2) There is only a finite number of elements in $\pi(DT(\mathcal{G}_F\mathcal{P}))$. □

So far we know that if all simplices in $DT(\mathcal{G}_F\mathcal{P})$ are mapped as simplices onto $\mathbb{X}$, then the whole triangulation is mapped onto a set of simplices in $\mathbb{X}$. We now consider the incidence relation.

Observation 13 *Assume that the restriction of π to any simplex in $DT(\mathcal{G}_F\mathcal{P})$ is injective. If τ is a simplex in $\pi(DT(\mathcal{G}_F\mathcal{P}))$ and $\tau' \leq \tau$, then τ' is a simplex in $\pi(DT(\mathcal{G}_F\mathcal{P}))$. This follows immediately from the fact that incidence relations are maintained by π and from Lemma 11.*

It only remains to show condition (ii), i.e. the intersection of two simplices σ and τ in $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is another simplex χ that is incident to both σ and τ.

Lemma 14 *Assume that the restriction of π to any simplex in $DT(\mathcal{G}_F\mathcal{P})$ is injective. Let $\sigma,\tau \in \pi(DT(\mathcal{G}_F\mathcal{P}))$ be any two simplices in $\mathbb{X}$, then $\sigma\cap\tau$ is a set of simplices in $\pi(DT(\mathcal{G}_F\mathcal{P}))$.*

Proof. Without loss of generality, we assume that $\sigma\cap\tau \neq \emptyset$. We show that $\sigma\cap\tau = \bigcup_{p\in\sigma\cap\tau}\chi_p$, where χ_p is a simplex in $\pi(DT(\mathcal{G}_F\mathcal{P}))$. The union is finite because there are only finitely many simplices in $\pi(DT(\mathcal{G}_F\mathcal{P}))$. Consider a point $p \in \sigma\cap\tau$. If p is a vertex of $\pi(DT(\mathcal{G}_F\mathcal{P}))$, then it is not contained in the interior of any other simplex, according to Lemma 11, and we set $\chi_p = \{p\}$. If p is not a vertex in $\pi(DT(\mathcal{G}_F\mathcal{P}))$, then $p \in \mathring{\sigma}'$ and $p \in \mathring{\tau}'$ for some proper faces $\sigma' \leq \sigma$ and $\tau' \leq \tau$ because σ and τ are internally disjoint (Lemma 11). Since σ' and τ' are again either internally disjoint or identical, it follows that they are the same face and we set $\chi_p := \sigma' = \tau'$. By condition (i) the simplex χ_p is contained in $\pi(DT(\mathcal{G}_F\mathcal{P}))$. □

Remember that $|\mathrm{St}(v)|$ denotes the union of the simplices in the star of v. We can now formulate the following sufficient condition:

Lemma 15 *If for all vertices v of $DT(\mathcal{G}_F\mathcal{P})$ the restriction of the quotient map $\pi|_{|\mathrm{St}(v)|}$ is injective, then $\pi(DT(\mathcal{G}_F\mathcal{P}))$ forms a simplicial complex.*

Proof. We set $\mathcal{K} = \pi(DT(\mathcal{G}_F\mathcal{P}))$. Let σ be a simplex of $DT(\mathcal{G}_F\mathcal{P})$ and v an incident vertex. Then $\sigma \subseteq |\mathrm{St}(v)|$, thus the restriction of $\pi|_{|\mathrm{St}(v)|}$ to σ is injective as well, and $\mathcal{K}$ is a set of simplices (Lemma 11).

Conditions (i) and (iii) follow from the above discussion. It remains to show condition (ii): Consider two simplices $\sigma, \tau \in \mathcal{K}$ with $\sigma \cap \tau \neq \emptyset$. By definition of a simplex, there exist sets $\mathcal{P}_\sigma, \mathcal{P}_\tau$ in $\mathcal{G}_F\mathcal{D}$ such that $\sigma = \pi(\mathrm{Ch}(\mathcal{P}_\sigma))$ and $\tau = \pi(\mathrm{Ch}(\mathcal{P}_\tau))$. From Lemma 14, we know that $\sigma \cap \tau$ is a set of simplices in $\mathcal{K}$. So there exists a vertex $v \in \sigma \cap \tau$ and $\sigma, \tau \in \mathrm{St}(v)$. By assumption $\pi|_{|\mathrm{St}(v)|}$ is injective, so π is injective on σ and τ, and $\sigma \cap \tau = \pi(\mathrm{Ch}(\mathcal{P}_\sigma)) \cap \pi(\mathrm{Ch}(\mathcal{P}_\tau)) = \pi(\mathrm{Ch}(\mathcal{P}_\sigma \cap \mathcal{P}_\tau))$. Also, the restriction of $\pi|_{|\mathrm{St}(v)|}$ to $\mathrm{Ch}(\mathcal{P}_\sigma \cap \mathcal{P}_\tau)$ is injective. So from Definition 3, it follows that $\sigma \cap \tau$ is a simplex. Since $\sigma \cap \tau \subseteq \sigma, \tau$, we have $\sigma \cap \tau \leq \sigma, \tau$. □

We can now prove Theorem 7. We set $\mathcal{K} = \pi(DT(\mathcal{G}_F\mathcal{P}))$. From Lemma 11 and Observation 13, we know that $\mathcal{K}$ is a finite set of simplices that fulfills conditions (i) and (iii). Assume that $\mathcal{K}$ is not a simplicial complex. From Lemma 15 there is a vertex $v \in \mathcal{K}$ for which $\pi|_{|\mathrm{St}(v)|}$ is not injective. As π is continuous by definition, this implies the existence of two different points $p, q \in |\mathrm{St}(v)|$ with $\pi(p) = \pi(q)$. Let σ denote the simplex of $\mathcal{K}$ that contains $\pi(p) = \pi(q)$ in its interior. Then there are two different simplices $\sigma'_{\mathbb{R}} \in \pi^{-1}(\sigma)$ and $\sigma''_{\mathbb{R}} \in \pi^{-1}(\sigma)$ containing p and q, respectively. Thus $\sigma'_{\mathbb{R}}$ and $\sigma''_{\mathbb{R}}$ are both elements of $\overline{\mathrm{St}(v)}$ (the *closure* $\overline{\mathcal{L}}$ of a subset $\mathcal{L} \subseteq \mathcal{K}$ is the smallest subcomplex of $\mathcal{K}$ containing $\mathcal{L}$). Let u, w be vertices different from v with $u \leq \sigma'_{\mathbb{R}}$ and $w \leq \sigma''_{\mathbb{R}}$. The vertices u, w are also elements of $\overline{\mathrm{St}(v)}$ and thus there are edges (u, v) and (v, w) in $DT(\mathcal{G}_F\mathcal{P})$. From $\pi(\sigma'_{\mathbb{R}}) = \pi(\sigma''_{\mathbb{R}})$ follows that $\pi(u) = \pi(w)$, and so the projection of (u, v) and (v, w) under π forms a cycle of length two in $\mathbb{X}$, which contradicts the assumption that $\pi|_{|\mathrm{St}(v)|}$ is injective. So $\mathcal{K}$ must be a simplicial complex that is homeomorphic to $\mathbb{X}$ by Lemma 12, which means that $\pi(DT(\mathcal{G}_F\mathcal{P}))$ is a triangulation of $\mathbb{X}$.

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